MINISTRY OF EDUCANION OF UKRAINE
NATIONAL MINING ACADEMY

THEORETICAL MECHANICS

KINEMATICS

Summary of Lectures

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CONTENTS

1. KINEMATICS OF A PARTICLE

1.1. Introduction to Kinematics

Kinematics is a part of mechanics which treats of the geometrical aspects of the bodies, without taking into account their inertia, i.e., mass or the forces acting on them.

Kinematics is an introduction to dynamics, insofar as the fundamental concepts and relationships of Kinematics have to be understood before studying the motion of bodies taking into account the action of forces. On the other hand, the methods of Kinematics are in themselves of practical importance, for example in studying the transmission of motion in mechanisms.

By motion in Kinematics is meant the relative displacement with time of a body in space with respect to other bodies.

In order to locate a moving body or particle we assume a coordinate system, which we call the frame of reference or reference system, to be fixed relative to the body with respect to which the motion is being considered. If the coordinates of all the points of a body remain constant within a given frame of reference, the body is said to be at rest relative to that reference system. If, on the other hand, the coordinates of any points of the body change with time, the body is said to be in motion relative to the given frame of reference.

Any motion in space takes place with time. In mechanics we deal with three - dimensional Euclidean space. Time in mechanics is considered as universal, i.e., as passing simultaneously in all our frames of reference. Time is continuously varying quantity. In problems of Kinematics, time is taken as an independent variable or argument. All other variables are regarded as changing with time, i.e., as functions of time. Any given instant of time is specified by the number of seconds that has passed between the initial and the given time. The difference between successive instants of time is called the time interval.

The principles of Kinematics, evolved from and confirmed by practical experience, are based on the axioms of geometry. No other laws or axioms are necessary for the Kinematics study of motion.

For the solution of problems of Kinematics, the motion under consideration has to be described. To Describe the motion, or the law of motion, of a given body means to specify the position of that body relative to a given frame of reference for any moment of time.

The principal problem of Kinematics is that of determining all the Kinematics characteristics of the motion of a body as a whole or of any of its particles (path, velocity, acceleration, etc.) when the law of motion is known.

We shall start the study of Kinematics with an investigation of the simplest object - a particle, proceeding later to the examination of the Kinematics of rigid body.

1.2. Methods of Describing Motion of a Particle

To describe the motion of a particle, it is necessary to specify its position in chosen frame of reference at any given time. There are three methods of describing motion: the natural method, the coordinate method and the vector one. Let us consider all of them.

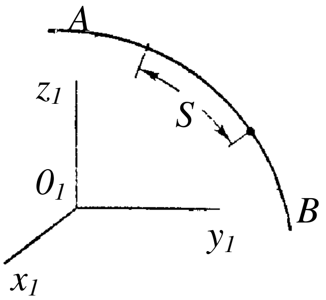


Fig.1.

I) Natural method of Describing Motion.

The continuous curve described by a particle moving with respect to a given frame of reference is called the path of that particle. If the path is a straight line, the motion is said to be rectilinear, if the path is a curve, the motion is curvilinear.

Let the curve *AB* in Fig.1 be the path of a particle *M* moving with respect to a frame of reference *OXYZ*. Take any fixed point *0* on the path as the origin of another frame of reference. Taking the path as an arc-coordinate axis, assume the positive and negative directions, as is done with rectangular axes. The position of the particle *M* on the path is now specified by a single coordinate *S*, equal to the distance from *0* to *M* measured along the arc of the path and taken with the appropriate sign. The distance *S* changes with time. In order to know the position of *M* on the path at any instant, we must know the relation:

$$S = S(t) \tag{1}$$

Eq. (I) expresses the law of motion of particle *M* along its path.

Thus, in order to describe the motion of a particle by the natural method, a problem must state the path of the particle, the origin on the path, showing the positive and negative directions, the equation of the particle's motion along the path in the form (l). Note that *S* in Eq. (I) denotes the moving particle's position, not the distance traveled by it. The Natural method is convenient when the particle's path is known at once.

2) Coordinate Method of Describing Motion.

A particle's path may not be known, that is why the coordinate method is employed more frequently.

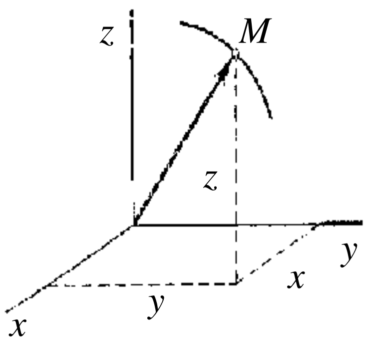


Fig.2

The position of a particle with respect to a given frame of reference *OXYZ* can be specified by its Cartesian coordinates *x, y, z* (Fig.2). If we want to know the equation of motion of a particle, i.e. its location in space at any instant, we must know its coordinates for any moment of time:

$$x = x(t), \quad y = y(t), \quad z = z(t). \tag{3}$$

Eqs. (3) are the equations of motion of a particle in terms of the Cartesian rectangular coordinates. They describe the curvilinear motion of a particle by the coordinate method.

It's obvious, that if a particle moves in one plane, then, we shall obtain two equations of motion.

Eqs (3) are, at the same time, the equations of the particle's path in parametric form. By eliminating time t from the equations of motion we can obtain the equation of the path in the usual form.

3) Vector Method of Describing Motion.

Let a particle M be moving relative to any frame of reference $OXYZ$. The position of the particle at any instant can be specified by a vector $\bar{r}$ drawn from the origin O to the particle M. Vector $\bar{r}$ is called the radius vector of the particle M.

When the particle moves, the vector $\bar{r}$ changes with time in both magnitude and direction. Thus, $\bar{r}$ is a variable vector (a vector function) depending on the argument t :

$$\bar{r} = \bar{r}(t). \tag{4}$$

Eq.(4) describes the curvilinear motion of a particle in vector form and can be used to construct a vector $\bar{r}$ for any particular moment of time and to determine the position of the moving particle at that moment.

The locus of the tips of vector $\bar{r}$ defines the path of the moving particle.

The vector method is convenient for establishing general dependencies, as it describes a particle's motion in terms of one vector equation (4) instead of the three scalar equations (3).

1.3 . Velocity of a Particle

One of the basic kinematical characteristics of motion of a particle is a vector quantity called velocity.

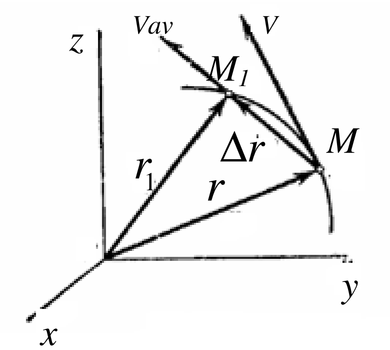


Fig.3

Let a moving particle occupy at time t a position M defined by the radius vector $\bar{r}$, and at time t_1 a position M_1 defined by the radius vector $\bar{r}$ (Fig.3). The displacement during the time interval $\Delta t = t_1 - t$ is defined by a vector $\overline{MM_1}$ which we shall call the displacement vector of the particle.

From triangle OMM_1 we obtain $\bar{r} + \overline{MM_1} = \bar{r}_1$, whence $\overline{MM_1} = \bar{r}_1 - \bar{r} = \Delta\bar{r}$. The ratio $\bar{V}_{av} = \dfrac{\overline{MM_1}}{\Delta t} = \dfrac{\Delta\bar{r}}{\Delta t}$ is called the average velocity of the particle during the given time interval Δt. Obviously, the smaller the time interval $\Delta t = t_1 - t$ the more precisely the average velocity will characterize the particle's motion. To obtain a characteristic of motion independent of the choice of the time interval the concept of instantaneous velocity is introduced. It is a vector quantity $\bar{V}$ towards which the average velocity $\bar{V}_{av}$ tends when the time interval Δt tends to zero:

$$\bar{V} = \lim_{\Delta t \to 0}\left(\bar{V}_{ab}\right) = \lim_{\Delta t \to 0}\frac{\Delta \bar{r}}{\Delta t} = \frac{d\bar{r}}{dt}.$$

Thus, the vector of instantaneous velocity of a particle is equal to the first derivative of the radius vector of the particle with respect to time:

$$\bar{V} = \frac{d\bar{r}}{dt}. \tag{5}$$

As the limiting direction of the secant MM_1 is a tangent, the vector of instantaneous velocity is tangent to the path of the particle in the direction of motion.

1.4. Acceleration of a Particle

Acceleration characterizes the time rate of change of velocity in magnitude and direction.

Let a moving particle have a velocity $\bar{V}$ at a given time t, and a velocity $\bar{V}_1$ - at time t_1. The increase in velocity in the time interval $\Delta t = t_1 - t$ is $\Delta\bar{V} = \bar{V}_1 - \bar{V}$. The ratio of the velocity increment vector $\Delta\bar{V}$ to the corresponding time interval $\Delta\, t$ defines the vector of average acceleration of the particle in the given time interval:

$$\bar{a}_{av} = \frac{\Delta\bar{V}}{\Delta t}.$$

The instantaneous acceleration at a given time t is defined as the vector quantity $\bar{a}$ towards which the average acceleration $\bar{a}_{av}$ tends when the time interval Δt tends to zero:

$$\bar{a}_{av} = \lim_{\Delta t \to 0}\frac{\Delta\bar{V}}{\Delta t} = \frac{d\bar{V}}{dt},$$

or, taking into account (5),

$$\bar{a} = \frac{d\bar{r}}{dt} = \frac{d^2\bar{V}}{dt^2}. \tag{6}$$

Hence, the instantaneous acceleration of a particle is equal to the first derivative of the velocity vector or the second derivative of the radius vector of the particle with respect to time.

In general case, the acceleration vector $\bar{a}$ lies in the osculating plane and is directed towards the inside of the curve.

The osculating plane trough a point M on a curve may be defined as the limiting position of a plane trough points M, M_1 and M_2 of the given curve when points M.

The osculating plane of a plane curve is coincident with the plane of the curve and is common for all its points.

1.5. Determination of the Velocity and acceleration of a Particle when its Motion is Described by the Coordinate Method

The following well-known theorem will be found useful in solving vector equations containing derivatives when it is necessary to go over from relations between vectors to relations between their projections:

the projection of the derivative of a vector on a fixed axis is equal to the derivative of the projection of the differentiated vector on the same axis.

That is to say, if $\overline{p} = \overline{i}p_x + \overline{j}p_y + \overline{k}p_z$,

$\overline{q} = \overline{i}q_x + \overline{j}q_y + \overline{k}q_z$ and $\overline{q} = \dfrac{d\overline{p}}{dt}$, then

$$q_x = \frac{dp_x}{dt}, \qquad q_y = \frac{dp_y}{dt}, \qquad q_z = \frac{dp_z}{dt}. \tag{7}$$

Let the motion of the particle be described by Eqs, (3). The velocity vector of a particle is $\overline{V} = \dfrac{d\overline{r}}{dt}$.

Using the relations (7) we obtain:

$$V_x = \frac{dx}{dt}, \qquad V_y = \frac{dy}{dt}, \qquad V_z = \frac{dz}{dt},$$

or

$$V_x = \dot{x}, \qquad V_y = \dot{y}, \qquad V_z = \dot{z}. \tag{8}$$

where *x, y* and *z* are the coordinate of the particle and the dot over the letter is a symbol of differentiation with respect to time.

Thus the projections of the velocity on the coordinate axes are equal to the first derivatives of the corresponding coordinates of the particle with respect to time.

Knowing the projections of the velocity we can find its magnitude and direction from the equations:

$$V_x = \sqrt{V_x^2 + V_y^2 + V_z^2}, \tag{9}$$

$$\cos\alpha = \frac{V_x}{V} \qquad \cos\beta = \frac{V_y}{V} \qquad \cos\gamma = \frac{V_z}{V}.$$

The acceleration vector of a particle is $\overline{a} = \dfrac{d\overline{V}}{dt}$.

Hence, from the theorem of the projection of a derivative and from Eqs. (8), we obtain:

$$a_x = \frac{dV_x}{dt} = \frac{d^2x}{dt^2}, \quad a_y = \frac{dV_y}{dt} = \frac{d^2y}{dt^2}, \quad a_z = \frac{dV_z}{dt} = \frac{d^2x}{dt^2} \tag{10}$$

or

$$a_x = \dot{V}_x = \ddot{x}, \quad a_y = \dot{V}_y = \ddot{y}, \quad a_z = \dot{V}_z = \ddot{z}.$$

Thus, the projections of the acceleration on the coordinate axes are equal to the first derivatives of the projections of the velocities, or the second derivatives of the corresponding coordinates of the particle with respect to time.

The magnitude and direction of the vector acceleration are given by the equations:

$$a = \sqrt{a_x^2 + a_y^2 + a_z},\qquad (11)$$

$$\cos\alpha_1 = \frac{a_x}{a},\qquad \cos\beta_1 = \frac{a_y}{a},\qquad \cos\gamma_1 = \frac{a_z}{a},$$

where α_1, β_1 and γ_1 are the angles made by the acceleration vector with the coordinate axes.

1.6. Determination of the Velocity of a Particle when its Motion is Described by the Natural Method

Given the path of a particle and the law of motion along it in the form $S=S(t)$.

Let us see how the velocity of a particle can be determined. If in a time interval $\Delta t=t_1-t$ a particle moves from position M to position M_1, displacement along the arc of the path being $\Delta S=S_1-S$ (Fig.4), the numerical value of the average velocity will be: $V_{av} = \dfrac{S_1 - S}{t_1 - t} = \dfrac{\Delta S}{\Delta t}$.

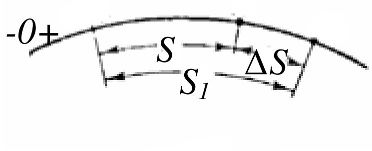


Fig.4

Passing to the limit we obtain the numerical value of the instantaneous velocity for a given time t:

$$V = \lim_{\Delta t\to 0.} \frac{\Delta S}{\Delta t}\qquad V = \frac{dS}{dt} = \dot{S}.\qquad (12)$$

Hence, the numerical value of the instantaneous velocity of a particle is equal to the first derivative of the displacement of the particle with respect to time.

The velocity vector is tangent to the path, the latter assumed to be known.

1.7. Determination of the Acceleration of a Particle When its Motion is Described by the Natural Method

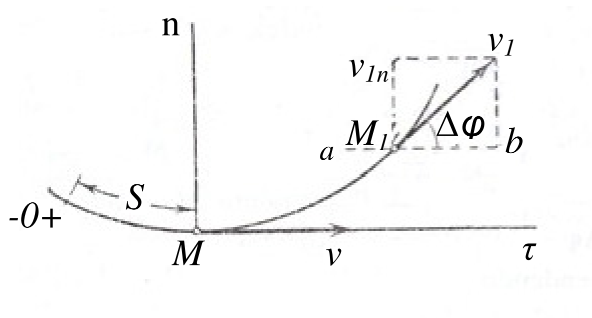


Fig.5

In p. 1.5 we considered the stationary coordinate axes. In the natural method of describing motion, vector a is determined from its projections on a set of coordinate axes $M\tau nb$ whose origin is at M and who moves together with the body (Fig.5).

These axes called the axes of a natural trihedron or velocity axes are directed as follows: axis $M\tau$ along the tangent to the path in the direction of the positive displacement S, axis Mn along the normal in the osculating plane towards the inside of the path, and axis Mb perpendicular to the former two to form a right-hand set. The normal *Mn*, which lies in the osculating plane, is called the principal normal, and the normal *Mb* perpendicular to it is called the binormal.

The projection of vector $\bar{a}$ on the binormal is zero $\left(a_b = 0\right)$, as vector acceleration lies in the osculating plane and the axis *Mb* is a perpendicular to it.

Let us determine the projections of $\bar{a}$ on the other two axes. Let the particle occupy a position *M* and have a velocity $\bar{V}$ at any time *t*, and at time $t_1 = t+\Delta t$ let it occupy a position M_1 and have a velocity $\bar{V}_1$. Then, by virtue of the definition,

$$\bar{a} = \lim_{\Delta t \to 0} \frac{\Delta \bar{V}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\bar{V}_1 - \bar{V}}{\Delta t}.$$

Let us project both parts of this equation on the axes $M\tau$ and *Mn* through point *M* taking into account the theorem of the projection of a vector sum on an axis. Then we obtain:

$$a_\tau = \lim_{\Delta t \to 0} \frac{V_{1\tau} - V_\tau}{\Delta t}, \qquad a_n = \lim_{\Delta t \to 0} \frac{V_{1n} - V_n}{\Delta t}.$$

Denote the angle between the direction of vector $\bar{V}_1$ and $\bar{V}$ by the $\Delta\varphi$. This angle is called the angle of contiguity.

It will be recalled that the limit of the ratio of the angle of contiguity $\Delta\varphi$ to the arc $\overline{MM_1} = \Delta S$ defines the curvature *K* of the curve at point *M*. As the curvature is the inverse of the radius of curvature ρ at *M*, we have:

$$\lim_{\Delta S \to 0} \frac{\Delta\varphi}{\Delta S} = K = \frac{1}{\rho}.$$

From Fig. 5 we see: $V_\tau = V$, $\quad V_n = 0,\qquad V_{1\tau} = V_1 \cos\Delta\varphi,\qquad V_{1n} = V_1 \sin\Delta y.$

Hence,

$$a_\tau = \lim_{\Delta t \to 0} \frac{V_1 \cos\Delta\varphi - V}{\Delta t}, \quad a_n = \lim_{\Delta t \to 0} \left(V_1 \frac{\sin\Delta\varphi}{\Delta t} \right). \tag{13}$$

It is obvious, that when $\Delta t \to 0$, point M_1 approaches *M* indefinitely, and simultaneously $\Delta\varphi \to 0, \Delta S \to 0, V_1 \to V$.

Hence, taking into account that $\lim\limits_{\Delta\varphi \to 0} \cos\left(\Delta\varphi\right) = 1$ we obtain:

$$a_\tau = \lim_{\Delta t \to 0} \frac{V_1 - V}{\Delta t} = \frac{dV}{dt}.$$

Multiplying the numerator and denominator of the second formula in Eq.(13) by $\Delta\varphi\Delta S$ we find:

$$a_n = \lim_{\Delta t \to n} \left(V_1 \frac{sin\Delta\varphi}{\Delta\varphi} \times \frac{\Delta\varphi}{\Delta S} \times \frac{\Delta S}{\Delta t} \right) = \frac{V^2}{\rho},$$

since, when $\Delta t \to 0$ the limits of each of the cofactors inside the brackets are as follows:

$$\lim V_1 = V, \quad \lim \frac{sin\Delta\varphi}{\Delta\varphi} = 1, \quad \lim \frac{\Delta\varphi}{\Delta S} = \frac{1}{\rho}, \quad \lim \frac{\Delta S}{\Delta t} = \frac{dS}{dt} = V.$$

Finally we obtain:

$$a_\tau = \frac{dV}{dt} = \frac{d^2S}{dt^2}, \quad a_n = \frac{V^2}{\rho}. \tag{14}$$

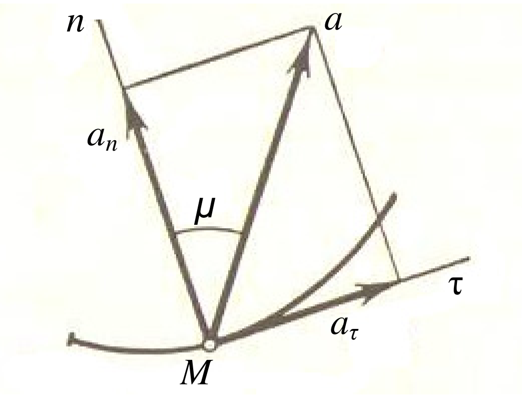


Fig.6

Thus, the projection of the acceleration of a particle on the tangent to the path is equal to the first derivative of the numerical value of the velocity, or the second derivative of the displacement with respect to time; the projection of the acceleration on the principal normal is equal to the second power of the velocity divided by radius of curvature of the path at the given point of the curve.

Lay off vectors $\bar{a}_\tau$ and $\bar{a}_n$ along the tangent $M\tau$ and the principal normal Mn. The component $\bar{a}_n$ is always directed along the inward normally as $\bar{a}_n$ > *0*, while the component a_τ can be directed either in the positive or in the negative direction of the axis $M\tau$ depending on the sign of the projection $\bar{a}_\tau$.

As the components $\bar{a}_\tau$ and $\bar{a}_n$ are mutually perpendicular y the magnitude of vector $\bar{a}$ and its angle μ to the normal Mn are given by the equations:

$$a = \sqrt{a_\tau^2 + a_n^2}, \quad tan\,\mu = \frac{|a_\tau|}{a_n}. \tag{15}$$

Thus, from Eqs. (I2) and (15) we can determine the magnitude and direction of the velocity and acceleration of the particle for any instant.

1.8. Some Special Cases of Particle Motion

Let us investigate some special cases of particle motion.

1) Rectilinear Motion. If the path of a particle is a straight line, then $\rho=\infty$, $a_n = \frac{V^2}{\rho} = 0$ and the tota1 acceleration is equal to the tangential acceleration:

$$a = a_\tau = \frac{dV}{dt}.$$

As in this case the velocity changes only in magnitude, we conclude that the tangential acceleration, characterizes the change of speed.

2) Uniform Curvilinear Motion. Curvilinear motion is uniform when the speed is constant: V = *const*. Then $a_\tau = \frac{dV}{dt} = 0$, and the total acceleration is equal to the

normal one: $a = a_n = \dfrac{V^2}{\rho}$. In this case vector $\overline{a}$ is continuously directed along the normal to the path of the particle.

As in this case acceleration is represented only by the change in the direction of the velocity, we conclude that the normal acceleration characterizes the change of the velocity in direction.

Let us deduce the equation of uniform curvilinear motion. From Eq.(12) we have $dS = Vdt$.

Let a particle be at the initial time $t = 0$ at a distance S_0 from the origin. Integrating both members of the equation over the respective intervals we obtain:

$$\int_{S_0}^{S} dS + \int_0^t Vdt \quad \text{or} \quad S - S_0 = Vt.$$

Finally, we obtain the equation of uniform curvilinear motion in the form:

$$S = S_0 + Vt.$$

3) Uniform Rectilinear Motion. In this case obviously, $a_n = a_\tau = 0$, therefore, $a = 0$. Note that uniform rectilinear motion is the only case of motion in which the acceleration is continually zero.

4) Uniformly Variable Curvilinear Motion. Curvilinear motion is called uniformly variable if the tangential acceleration is constant: $a_\tau = const$

Let us deduce the equation of this motion assuming that at $t=0$, $S=S_0$ and $V=V_0$. By Eq.(I4), $dV = a_\tau dt$.

As $a_\tau = const$ integrating both members of the last equation over the corresponding intervals gives us:

$$V = V_0 + a_\tau t.$$

Let us write this equation in the form:

$$\frac{dS}{dt} = V_0 + a_\tau t, \quad \text{or} \quad dS = V_0 dt + a_\tau t dt.$$

Integrating again we obtain:

$$S = S_0 + V_0 t + a_\tau \frac{t^2}{2}.$$

If, in curvilinear motion, the speed increases, the motion is said to be accelerated, if it decreases the motion is said to be retarded.

As the change in magnitude of the velocity is characterized by the tangential acceleration, the motion is accelerated if V and a_τ have the same signs and retarded if the signs are different.

In the particular case of uniformly variable motion, if V and a_τ are of the same sign the motion is uniformly accelerated, if they are of opposite sign the motion is uniformly retarded.

2. TRANSLATIONAL AND ROTATIONAL MOTION OF A RIGID BODY

2.1. Translational Motion

In Kinematics we shall regard all solids as rigid bodies, i.e., we shall assume that the distance between any two points of a body remains the same during the whole period of motion.

Problems of Kinematics are basically of two types: definition of the motion and analysis of the Kinematic Characteristics of the motion of a body as a whole and analysis of the motion of every point of the body in particular.

We shall begin with the consideration of the simplest motion, i.e., translational one.

Translation of a rigid body is such a motion in which any straight line through the body remains continually parallel to itself.

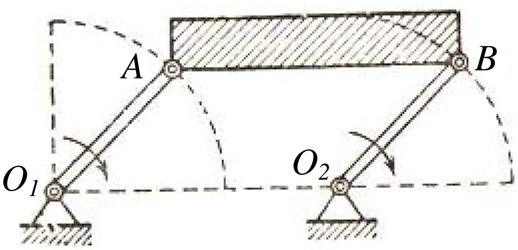


Fig.7

Translation should not be confused with rectilinear motion. In translation the particles of a body may move on any curved paths. Here is one example of translation.

The motion of the connecting rod AB in Fig.7 is that of translation, since, when the cranks O_1A and O_2B ($O_1A=O_2B$) rotate, any straight line through the rod remains parallel to itself. The particles of the connecting rod travel in circles.

The properties of translational motion are defined by the following theorem: in translational motion, all the particles of a body move along similar paths (which will coincide if superimposed) and have at any instant the same velocity and acceleration. Let us prove the theorem. Take two arbitrary points A and D on the body whose positions at time t are specified by radius vectors $\bar{r}_A$ and $\bar{r}_B$ (Fig. 8).

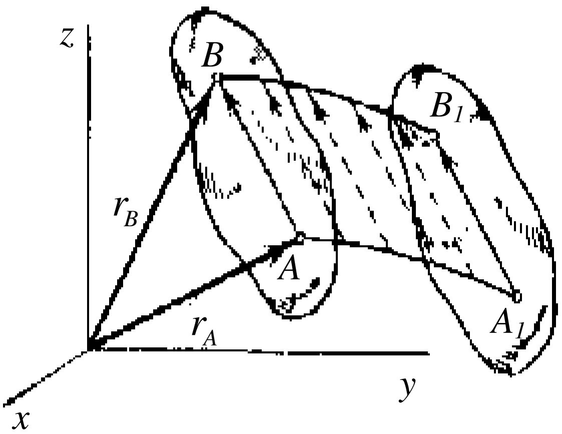


Fig.8

Draw a vector ***AB*** joining the two points. It is obvious that

$$\bar{r}_B = \bar{r}_A + \overline{AB}. \tag{16}$$

The length of $\overline{AB}$ is constant, being the distance between two points of a rigid body, and the direction of AB is constant by virtue of the translational motion of the body. Thus, the vector $\overline{AB}$ is constant. It follows then from Eq. (l6) that the path of particle B can be obtained by a parallel displacement of all the points of the path of particle A through a constant vector $\overline{AB}$.

Hence, the paths of particles A and B are identical curves which will coincide if superimposed.

Differentiating both parts of Eq. (I6) with respect to time we have:

$$\frac{d\bar{r}_B}{dt} = \frac{d\bar{r}_A}{dt} + \frac{d\left(\overline{AB}\right)}{dt}.$$

Taking into account that $\overline{AB}$ = *const* we obtain $\bar{V}_A = \bar{V}_B$, i.e., at any instant the velocities of points *A* and *B* are equal in magnitude and directions.

Differentiating again we obtain:

$$\frac{d\bar{V}_A}{dt} = \frac{d\bar{V}_B}{dt} \quad \text{or} \quad \bar{a} = \bar{a}_B.$$

Hence, at any instant the accelerations of *A* and *B* are equal in magnitude and direction.

As points *A* and *B* are arbitrary, it follows that the paths, the velocities and accelerations of all the points of a body at any instant are the same, which proves the theorem.

It follows from the theorem that the translational motion of a rigid body is fully described by the motion of any point belonging to it. Thus, the analysis of translational motion of a rigid body is reduced to the methods of particle Kinematics examined before.

2.2. Rotational Motion of a Rigid Body. Angular Velocity and Angular Acceleration

Rotation of a rigid body is such a motion in which there is always two points of the body or two points continuously connected with this body which remain motionless. The line through these fixed points is called the axis of rotation.

Since the distance between the points of a rigid body does not change, it is evident that all points of the axis of rotation are motionless, while all the other points of the body describe circular paths the plane of which are perpendicular to the axis of rotation and the centers of which lie on it.

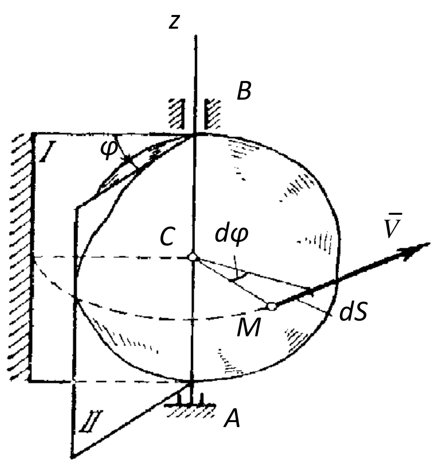


Fig.9

To determine the position of a rotating body, let us pass two planes through the axis of rotation *Az*; plane *I*, which is fixed, and plane *II* through the rotating body and rotating with it. The position of the body at any instant will be fully specified by the angle φ between the two planes, taken with the appropriate sign, which we shall call the angle of rotation of the body.

We shall consider the angle positive if it is laid off counterclockwise from the fixed plane by an observer looking from the positive end of axis Az and negative if it is laid off clockwise.

Then, the position of a body at any instant is completely specified if we know the angle φ as a function of time t, i.e.,

$$\varphi = \varphi(t). \tag{17}$$

Eq. (I7) describes the rotational motion of a rigid body. The basic Kinematical Characteristics of the rotation of a rigid body are its angular velocity ω and angular acceleration ε.

If in an interval of time $\Delta t = t_1 - t$ a body turns through an angle $\Delta\varphi = \varphi_1 - \varphi$, the average angular velocity of the body in the given time interval is $\omega_{av} = \dfrac{\Delta\varphi}{\Delta t}$.

The angular velocity of the body at a given time t is the in value towards which ω_{av} tends when the time interval Δt tends to zero:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} \quad \text{or} \quad \omega = \frac{d\varphi}{dt}. \tag{18}$$

Thus, the angular velocity of a body at a given time is equal in magnitude to the first derivative of the angle of rotation with respect to time.

The sign of ω specifies the direction of the rotation. It will be noticed that $\omega > 0$ when the rotation is counterclockwise, and $\omega < 0$ when it is clockwise.

The angular velocity of a body can be denoted by a vector $\bar{\omega}$ of magnitude $\omega = \dfrac{d\varphi}{dt}$ along the axis of rotation in the direction from which the rotation is seen as counterclockwise. Such a vector simultaneously gives the magnitude of the angular velocity, the axis of rotation, and. the sense of rotation about that axis.

Angular acceleration characterizes the time rate of change of the angular velocity of a rotating body.

If in time interval $\Delta t = t_1 - t$ the change of angular velocity of a body is $\Delta\omega = \omega_1 - \omega$, the average angular acceleration in that interval of time is

$$\varepsilon_{av} = \frac{\Delta\omega}{\Delta t}.$$

The angular acceleration at a given time is the value towards which ε_{av} tends when the time interval Δt tends to zero:

$$\varepsilon = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt},$$

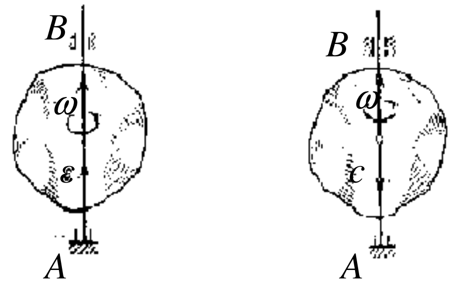


Fig.10

or taking into account Eq.(l8)

$$\varepsilon = \frac{d\omega}{dt} = \frac{d^2\varphi}{dt^2}. \tag{19}$$

Thus, the angular acceleration of a body at a given time is equal in magnitude to the first derivative of the angular velocity, or the second derivative of the angular displacement, of the body with respect to time.

If the angular velocity increases in magnitude, the rotation is accelerated, if it decreases, the rotation is retarded. It will be readily noticed that the rotation is accelerated when ω and ε are of the same sign, and retarded when they are of different sign.

By analogy with angular velocity, the angular acceleration of a body can be denoted by a vector $\bar{\varepsilon}$ along the axis of rotation. The direction of $\bar{\varepsilon}$ coincides with that of $\bar{\omega}$ when the rotation is accelerated, and is of opposite sense when the rotation is retarded.

2.3. Uniform and Uniformly Variable Rotations

If the angular velocity of a rotating body does not change the rotation is said to be uniform.

Let us develop the equation of uniform rotation. We have from Eq. (18) $d\varphi = \omega dt$. Hence, assuming that at the initial moment $t = 0$ angle $\varphi = 0$ and integrating the left-hand member from 0 to t and the right-hand member from 0 to t we obtain:

$$\varphi = \omega t. \tag{20}$$

In engineering, the velocity of uniform rotation is often expressed as the number of revolutions per minute (rpm).

Let us establish the relation between n rpm and $\omega\ sec^{-1}$. A complete revolution turns a body through an angle of 2π and n revolutions take it through an angle $2\pi n$. If the duration of this rotation is $t = 1\min = 60 sec$, then from Eq.(18) we have:

$$\omega = \frac{\pi n}{30} \approx 0{,}1n.$$

If the angular acceleration of a body does not change during the rotation $(\varepsilon = const)$, rotation is said to be uniformly variable.

Let us develop the equation of this motion assuming that at the initial instant t=0 angle φ=0 and that the angular velocity $\omega = \omega_0$.From Eq. (19) we have $d\omega = \varepsilon dt$. Integrating the both parts of this equation in respective limits we obtain:

$$\omega = \omega_0 + \varepsilon t. \tag{21}$$

Let us write Eq.(2I) in the form:

$$\frac{d\varphi}{dt} = \omega_0 + \varepsilon t \qquad \text{or} \qquad d\varphi = \omega_0 dt + \varepsilon t dt.$$

Integrating again we obtain the equation of uniformly variable rotation:

$$\varphi = \omega_0 t + \varepsilon \frac{t^2}{2}. \tag{22}$$

2.4. Velocities and Accelerations of the Points of a Rotating Body

Having established the characteristics of the motion of bodies as a whole, let us now investigate the motion of the individual points of a body.

Consider a point M of a rigid body at a distance h from the axis of rotation Az (Fig.9). If in time dt the body makes an infinitesimal displacement through the angle $d\varphi$, point M will have made displacement $dS = hd\varphi$ along its path. The velocity of the point is the ratio of dS to dt , i.e.,

$$V = \frac{dS}{dt} = h\frac{d\varphi}{dt}, \qquad \text{or} \qquad V = h\omega. \tag{23}$$

Thus, the linear velocity of a point belonging to a rotating body is equal to the product of the angular velocity of that body and the distance of the point from the axes of rotation.

The linear velocity is tangent to the circle described by point. As the value of ω at any given instant is the same for all points of the body, it follows from Eq. (23) that the liner velocity of any point of a rotating body is proportional to its distance from the axis of rotation (Fig. 11). In order to determine the acceleration of point, we apply equations

$$a_\tau = \frac{dV}{dt}, \quad a_n = \frac{V^2}{\rho}.$$

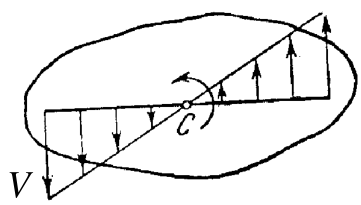


Fig. 11

In our case, $\rho = h$. Substituting the expression for V from Eq. (23) we obtain:

$$a_\tau = h\frac{d\omega}{dr}, \qquad a_n = \frac{h^2\omega^2}{h},$$

and finally: $a_\tau = h\varepsilon, \quad a_n = h\omega^2.$ (24)

The tangential and normal accelerations are shown in Fig.12. The total acceleration of a point is

$$a = \sqrt{a_\tau^2 + a_n^2} = h \cdot \sqrt{\varepsilon^2 + \omega^4}. \tag{25}$$

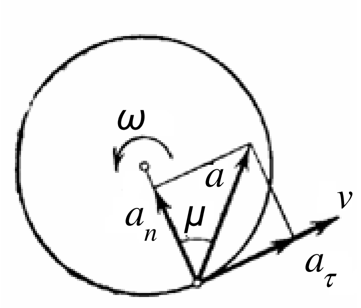


Fig.12

The inclination of the vector of total acceleration to the radius of the circle described by the point is specified, by the angle μ given by the equation:

$$tan\mu = \frac{|a_\tau|}{a_n} = \frac{|\varepsilon|}{\omega^2}. \tag{26}$$

Since at any given instant ε and ω are each the same for all the points of the body, it follows that the accelerations of all the points of a rotating body are proportional to their distances from the axis of rotation and make the same angle with the radii of the circles described by them.

3. PLANE MOTION OF A RIGID BODY

3.I. Equations of Plane Motion. Resolution of Motion Into Translation and Rotation

Plane motion of a rigid body is such motion in which all its points move parallel to a fixed plane. Many machine parts have plane motion, for example, a wheel running on a straight track or connecting rod of a reciprocating engine. Rotation is, in fact, special case of plane motion.

Let us consider the section *S* of a body produced by passing any plane *Oxy* parallel to a fixed plane *P* (Fig.13). All the points of the body belonging to a line MM' normal to plane *P* move in the

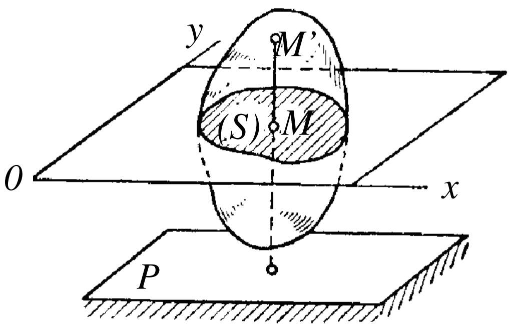


Fig. 13

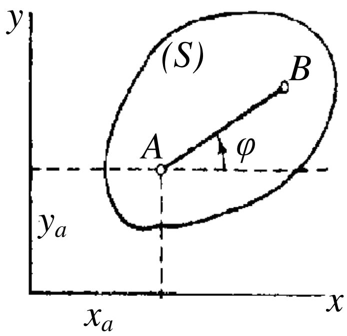


Fig. 14

same way. Therefore, in investigating plane motion it is sufficient to investigate the motion of section *S* of that body in the plan *Oxy*.

The position of section *S* in plane *Oxy* is completely specified by the position of any line *AB* in this section (Fig. I4) The position of the line *AB* may be specified by the coordinates x_A and y_A of point *A* and the angle φ between an arbitrary line *AB* in section *S* and axis *x*.

The point *A* chosen to define the position of section *S* is called the pole. As the body moves, the quantities x_A, y_A and φ will change with time and the motion of the body, i.e. its position in space at any moment of time, will be completely specified if we know

$$x_A = x_A(t), \quad y_A = y_A(t), \quad \varphi = \varphi(t). \tag{27}$$

Eqs.(27) are the equations of plane motion of a rigid body.

Consider the successive positions I and II of the section *S* of a moving body (Fig. 15). It will be observed that the following method can be employed to move section 0 from position I to position II. Let us first translate the body so that pole A_1 occupies

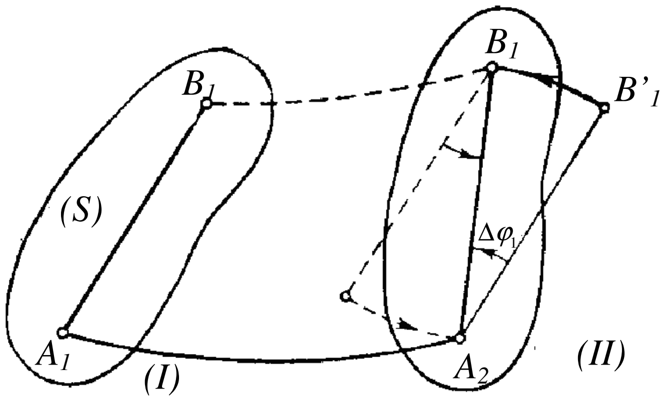


Fig. 15

position A_2 (line A_1B_1 occupies position A_2B_1') and then turn the section about pole A_2 through angle $\Delta\varphi_1$. In the same way we can move the body from position II to some new position III, etc.

Thus, the plane motion of a rigid body is a combination of a translation, in which all the points move in the same way as the pole A, and of a rotation about that pole.

The rotation takes place about an axis perpendicular to the plane P through the pole A. For the sake of brevity, however, we shall speak simply of rotation about the pole A.

The translational component of plane motion can, evidently, be described, by the first two of Eqs. (27), and the rotational component by the third one.

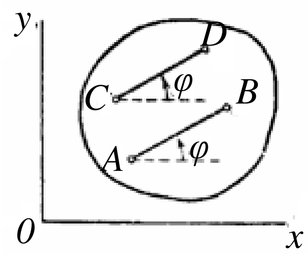


Fig.16

The principal kinematic characteristics of plane motion are the velocity and acceleration of translation, and the angular velocity ω and angular acceleration ε of the rotation about the pole. The values of these characteristics can be found for any time from Eqs.(27).

In analyzing plane motion, we are free to choose any point of the body as the pole. Let us consider a point C as a pole instead of A and determine the position of the line CD making an angle φ_1 with axis x (Fig.16). The characteristics of the translatory component of the motion would have been different, for in the general case $\bar{V}_C \neq \bar{V}_A$ and $\bar{a}_c \neq \bar{a}_a$ (otherwise the motion would be that of pure translation). The characteristics of the rotational component of the motion ω and ε remain, however, the same. For, drawing CB parallel to AB, we find that at an instant, of time $\varphi_1 = \varphi - \alpha_1$, where $\alpha = const$. Hence $\dot{\varphi}_1 = \dot{\varphi}$, $\ddot{\varphi}_1 = \ddot{\varphi}$, or $\omega_1 = \omega$, $\varepsilon_1 = \varepsilon$. Thus, the rotational component of motion does not depend, on the position of the pole.

3.2. Determination of the Path and Velocity of a Point of a Body

Let us now investigate the motion of individual points of a rigid body. We shall begin with the determination of the paths and velocities.

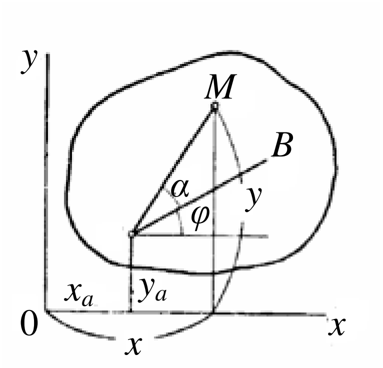


Fig.17

Consider a point M of a body whose position in the section S is specified by its distance $b = AM$ from the pole A and angle α (Fig. I7).If the motion, of the body is described by Eqs. (27), the x and y coordinates of point M in the system Oxy will be

$$x = x_A + b \cdot \cos(\varphi + \alpha),$$
$$y = y_A + b \cdot \sin(\varphi + \alpha), \qquad (28)$$

where x_A, y_A and φ are the functions of time given by Eqs. (27).

Eqs. (28) describe the motion of point M in plane Oxy and at the same time give the equation of the point's path in parametric form. The usual equation of the path can be obtained by eliminating time t from Eqs. (28).

As we know, plane motion of a rigid body is a combination of a translation in which all points of the body move with the velocity of the pole $\overline{V}_A$ and a rotation about that pole. Let us show that the velocity of any point M of the body is the geometrical sum of its velocities for each component of the motion.

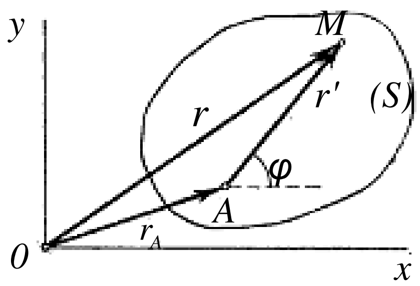


Fig. 18

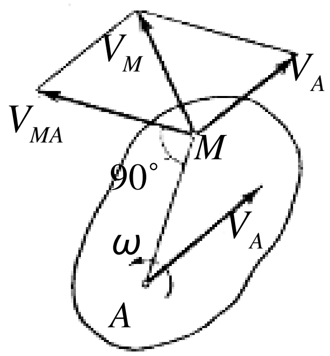


Fig.19

The position of a point M in section S is specified by the radius vector $\overline{r} = \overline{r}_A + \overline{r}'$ (Fig.18), where $\overline{r}_A$ is the radius vector of the pole A, $\overline{r}' = AM$ is the vector which specifies the position of point M with reference to the axes $Ax'y'$ that perform translational motion together with A. The motion of section S with reference to those axes is the motion about pole A. Then,

$$\overline{V}_M = \frac{d\overline{r}}{dt} = \frac{d\overline{r}_A}{dt} + \frac{d\overline{r}'}{dt}.$$

In this equation $\frac{d\overline{r}_A}{dt} = \bar{V}_A$ is equal to the velocity of pole A. This quantity $\frac{d\overline{r}'}{dt}$ is equal to the velocity $\bar{V}_{MA}$ of point M at $\overline{r}_A = const$, i.e., when A is fixed or, in other words, when the section S rotates about pole A. It thus follows from the preceding equation that

$$\bar{V}_M = \bar{V}_A + \bar{V}_{MA}. \qquad (29)$$

The velocity of rotation $\bar{V}_{MA}$ point M about pole A is $V_{MA} = \omega \cdot MA \left(\overline{V}_{MA} \perp MA \right)$, where ω is the angular velocity of the rotation of the body.

Thus, the velocity of any point of a body is the geometrical sum of the velocity of any other point taken as the pole and the velocity of rotation of point about the pole.The magnitude and direction of the velocity $\overline{V}_{MA}$ are found by constructing a parallelogram (Fig. 19).

3.3. Theorem of the Projections of the Velocities of Two Points of a Body. Instantaneous Centre of Zero Velocity

The use of Eq. (29) to determine the velocities of the points of a body usually leads to involved computations. However, we can evolve from Eq.(29) several simpler and more convenient methods of determining the velocity of any point of a body.

One of these methods is given by the theorem: the projections of the velocities of two points of a rigid body on the straight line joining those points are equal.

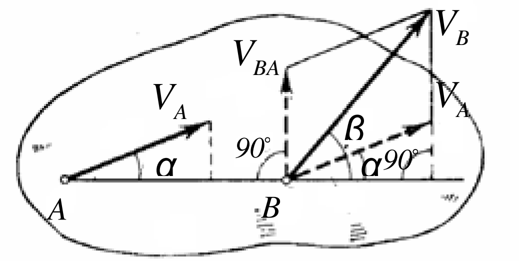


Fig.20

Consider any two points A and B of a body. Taking point A as the pole (Fig. 20) we have from Eq. (29) $\bar{V}_B = \bar{V}_A + \bar{V}_{BA}$.

Projecting both members of the equation_on AB and taking into account that vector V_{BA} is perpendicular to AB we obtain:

$$V_B \cos\beta = V_A \cos\alpha,$$

and the theorem is proved. This result offers a simple method of determining the velocity of any point of a body if the direction of motion of that point and the velocity of any other point of the same body are known.

Another simple and visual method of determining the velocity of any point of a body performing plane motion is based on the concept of instantaneous centre of zero velocity.

The instantaneous centre of zero velocity is a point belonging to the section of a body or its extension which at the given instant is momentarily at rest.

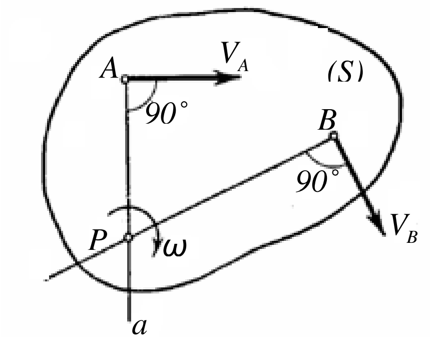


Fig.21

It will be readily noticed that if a body is in non-translational motion such one and only one point always exists at any instant. Let point A and B in section S of a body have, at time t, non-parallel velocities $\bar{V}_A$ and $\bar{V}_B$ (Fig. 21). Then point P of intersection of perpendiculars Aa, to vector $\bar{V}_A$ and Bb to vector $\bar{V}_B$ will be the instantaneous centre of zero velocity, as $\bar{V}_p = 0$. For, if we assumed that $\bar{V}_p \neq 0$ then, by the theorem of the projections of the velocities of the points of a body, vector $\bar{V}_p$ would have to be simultaneously perpendicular to AP (as $\bar{V}_A \perp AP$) and to BP (as $\bar{V}_B \perp BP$), which is impossible.

It also follows from the theorem that at the given instant, no other point of section S can have zero velocity (e.g., for point a, the projection of $\bar{V}_B$ on Ba is not zero and consequently $\bar{V}_a \neq 0$.

3.4. Determination of the Velocity of a Point of a Body Using the Instantaneous Centre of Zero Velocity

Using Eq. (29) and taking a point P as the pole at time t the velocity of point A will be

$$\bar{V} = \bar{V}_p + \bar{V}_{AP} = \bar{V}_{AP},$$

as $\bar{V}_p$ =0**.** The same result can be obtained for any other point of the body.

Thus, the velocity of any point of a body is equal to the velocity of its rotation about the instantaneous centre of zero velocity.

Hence, we have:

$$V_A = \omega \cdot PA\left(\bar{V}_A \perp PA\right),$$

$$V_B = \omega \cdot PB\left(V_B \perp PB\right), \text{ etc.} \tag{30}$$

It also follows from Eqs.(30) that

$$\frac{V_A}{P_A} = \frac{V_B}{PB}, \tag{31}$$

i.e., the velocity of any point of a body is proportional to its distance from the instantaneous centre of zero velocity.

These results lead to the following conclusions:

1) To determine the instantaneous centre of zero velocity, it is sufficient to know the directions of the velocities $\bar{V}_A$ and $\bar{V}_B$ of any two points *A* and *B* of a body or their paths. The instantaneous centre of zero velocity lies at the intersection of the perpendiculars erected from points *A* and *B* to their respective velocities, or to the tangents to their paths.

2) To determine the velocity of any point of a body, it is necessary to know the magnitude and direction of the velocity of any point *A* of that body and the direction of the velocity of another point *B* of the same body. Then, by erecting from points *A* and *B* perpendiculars to $\bar{V}_A$ and $\bar{V}_B$, we obtain the instantaneous centre of zero velocity *P* and, from the direction of $\bar{V}_A$, the sense of rotation of the body. Next, knowing $\bar{V}_A$ we can find from Eq. (3I) the velocity $\bar{V}_M$ of any point *M* of the body. Vector $\bar{V}_M$ is perpendicular to *PM* in the direction of the rotation.

3) The angular velocity of a body, as can be seen from Eqs. (30) is at any given instant equal to the ratio of the velocity of any point to its distance from the instantaneous centre of aero velocity *P*:

$$\omega = \frac{V_B}{PB}$$

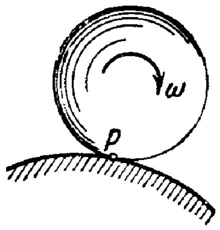
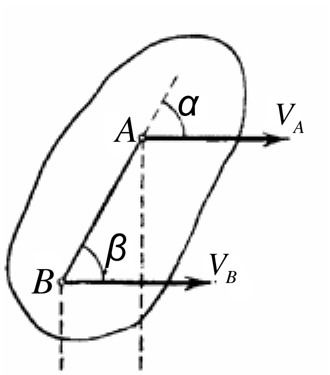


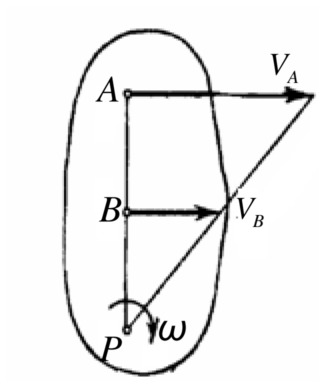
Fig.22

Let us consider some special cases of the instantaneous centre of zero velocity.

I) If plane motion is performed by a cylinder rolling without slipping along a fixed cylindrical surface, the point of contact *P* (Fig. 22) is momentarily at rest and, consequently, is the instantaneous centre of zero velocity ($\bar{V}_p = 0$ because if there is no slipping, the contacting points of both bodies must have the same velocity, and the second body is motionless). An example of such motion is that of a wheel running on a rail.

2) If the velocities of points *A* and *B* of the body are parallel to each other, and *AB* is not perpendicular to $\bar{V}_A$ (Fig. 23a)





a) *Fig. 23*

the instantaneous centre of zero velocity lies in infinity, and the velocities of all points are parallel to $\bar{V}_A$. Prom the theorem of the projections of velocities it follows that $V_A \cos\alpha = V_B \cos\beta$ i.e. $V_B = V_A$. The result is the same for all other points of the body. Consequently, in this case the velocities of all points of the body are equal in magnitude and direction at every instant, i.e., the instantaneous distribution of the velocities of the body is that of translation. This state of motion is also called instantaneous translation. The angular velocity of the body at the given instant is zero.

3) If the velocities of points *A* and *B* are parallel and *AB* is perpendicular to $\bar{V}_A$, the instantaneous centre of zero velocity *P* can be located by the construction shown in Fig. 23b. The validity of this construction follows from the proportion (3I).

3.5. Determination of the Acceleration of a Point of a Body

We shall demonstrate that, like velocity, the acceleration of any point of a body in plane motion is composed of its accelerations of translation and rotation. The location of point *M* with respect to axes *Oxy* (Fig. 18) is specified by the radius vector

$\bar{r} = \bar{r}_A + \bar{r}'$ where $\bar{r}' = \overline{AM}$. Hence,

$$\bar{a}_M = \frac{d^2 \bar{r}}{dt} = \frac{d^2 \bar{r}_A}{dt^2} + \frac{d^2 \bar{r}'}{dt^2}.$$

In this equation the quantity $\frac{d^2 \bar{r}_A}{dt^2} = \bar{a}_A$ is the acceleration of the pole *A*, and the quantity $\frac{d^2 \bar{r}'}{dt^2} = \bar{a}_{MA}$ is the acceleration of point *M* in its rotation with the body round *A*. Hence,

$$\bar{a}_M = \bar{a}_A + \bar{a}_{MA}. \qquad (32)$$

From Eqs. (25) and (26), the acceleration of point *M* in its rotation about *A* is

$$a_{MA} = MA\sqrt{\varepsilon^2 + \omega^4}, \qquad \tan \quad \mu = \frac{|\varepsilon|}{\omega^2},$$

where μ is the angle between the direction $\bar{a}_{MA}$ and line *MA*.

Thus, the acceleration of any point of a body is composed of the acceleration of any other point taken for the pole and the acceleration of the point in its rotation together with the body about that pole. The magnitude and direction of the acceleration $\bar{a}_M$ are determined by constructing a parallelogram (Fig. 24).

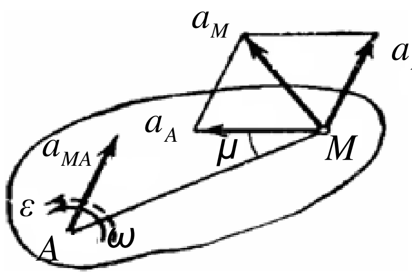


Fig.24

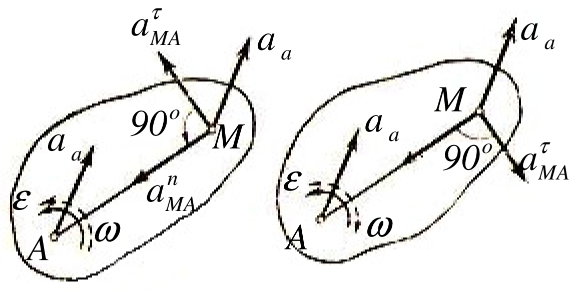


Fig.25

However, the computation of $\bar{a}_M$ by means of the parallelogram in Fig. 24 makes the solution more difficult, as it becomes necessary first to calculate the angle μ and then the angle between vectors a_{MA} and a_A. Therefore, in problem solutions it is more convenient to replace vector a_{MA} by its tangential and normal components:

$$a^\tau_{MA} = AM \cdot \varepsilon, \quad a^n_{MA} = AM \cdot \omega^2.$$

Vector $\bar{a}^\tau_{MA}$ is perpendicular to *AM* in the direction of the rotation if it accelerated, and opposite the rotation if it is retarded. Vector $\bar{a}^\tau_{MA}$ is always directed from point *M* to the pole *A* (Fig. 25).

Instead of Eq. (32) we obtain

$$\bar{a} = \bar{a}_A + \bar{a}^\tau_A + \bar{a}^n_{MA}. \tag{33}$$

If pole *A* is in non-rectilinear motion, its acceleration is also composed, of the tangential and normal accelerations, hence

$$\bar{a}_M = \bar{a}_{A\tau} + \bar{a}_{An} + \bar{a}^\tau_{MA} + \bar{a}^n_{MA}$$

4. RESULTANT MOTION OF A PARTICLE

4.1. Relative, Transport, and Absolute Motion

So far we have considered the displacement of a particle or body with respect to one given frame of reference. But in solving problems of mechanics it is often more expedient (and sometimes necessary) to consider the motion of a particle or body simultaneously with respect to two frames of reference, one of which is assumed to be fixed and the other moving in some specified way with reference to the first. The motion performed in this case is called resultant, or combined motion.

For example, when a sphere rolls on the deck of a moving boat, its motion with respect to the shore is the resultant of its rolling relative to the deck (the moving frame of reference) and its motion together with the deck with respect to the shore (the fixed frame of reference). Thus, the resultant motion of the sphere can be resolved into two simpler, and easier analyzed, motions. The method of resolving a motion into simpler motions by introducing a supplementary moving frame of reference is widely employed in kinematical calculations.

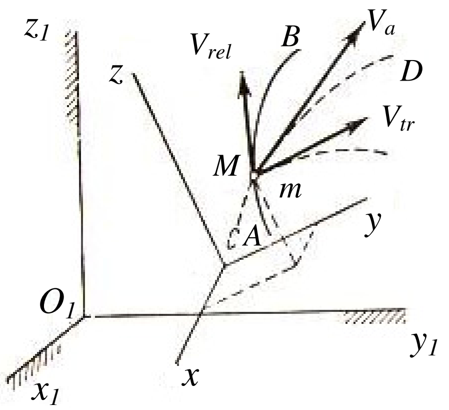


Fig.26

Consider the resultant motion of a particle M moving with respect to a frame of reference $Oxyz$ which is in turn moving with relation to another frame of reference $O_1\, x_1\, y_1\, z$, which we assume to be fixed (Fig. 26).We employ the following definitions.

The motion performed by the particle M with respect to the moving coordinate system is called relative motion. The path AB described by the particle in relative motion is called the relative path. The velocity of the motion of particle M relative to the axes ($Oxyz$, i.e, along the curve AB is called the relative velocity (denoted by the symbol $\overline{V}_{rel}$), and the particles acceleration in that motion is the relative acceleration (denoted $\overline{a}_{rel}$). It $\overline{a}_{rel}$ follows from the definition that in computing $\overline{V}_{rel}$ and axes $Oxyz$ can be assumed to be fixed.

The motion performed by the moving frame of reference $Oxyz$, together with all the points of space fixed with respect to it, relative to the fixed system $0_1\, x_1\, y_1\, z_1$ is for the particle M, the motion of transport.

The velocity of the point fixed in the moving axes $Oxyz$ with which the particle M coincides at a given instant is called the transport velocity of the particle M at that instant (denoted by $\overline{V}_{tr}$), and the acceleration of that point is called the transport acceleration of the particle M (denoted by $\overline{a}_{tr}$).

If we imagine the relative motion of particle M to be taking place on the surface or inside of a rigid body in which the moving coordinates $Oxyz$ are fixed, then the transport velocity or acceleration of particle M at any given instant is the velocity or acceleration of the point of the body which coincides with M at that instant.

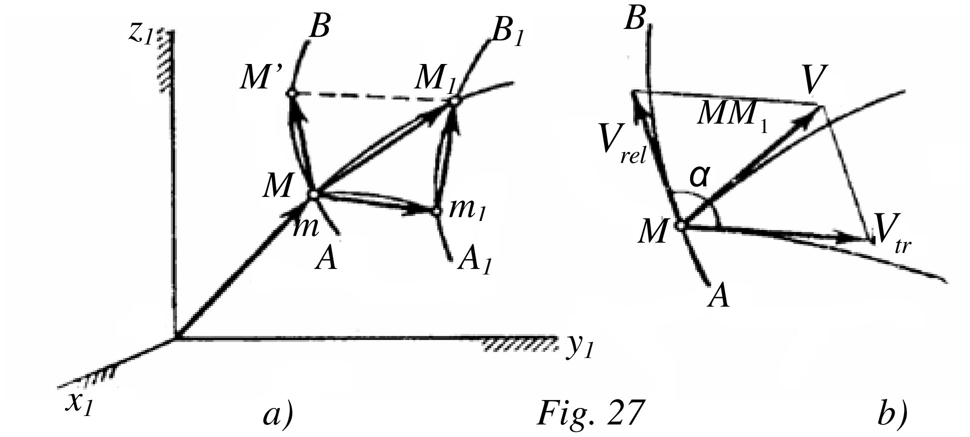
The motion of the particle with respect to the fixed frame of reference $O_1\, x_1\, y_1\, z_1$ is called the absolute, or resultant, motion. The path CD described in this motion is called the absolute path, the velocity, is the absolute velocity (denoted $\overline{V}_a$), and the acceleration, the absolute acceleration (denoted $\overline{a}_a$).

In the example cited in the beginning of this section, the motion of the sphere with respect to the deck is relative motion, and the velocity of this motion is the relative velocity of the sphere. The motion of the ship with respect to the shore is, for the sphere, the motion of transport, and the velocity of the point of the deck with which the sphere coincides at the given time is, for the sphere, the transport velocity. Finally, the motion of the sphere, with respect to the shore is the absolute motion of the sphere and the velocity of that motion is the absolute velocity of the sphere.

In order to solve the relevant problems of Kinematics, it is necessary to establish the relationships between the velocities and accelerations of the relative, transport, and absolute motions.

4.2. Composition of velocities

Let us consider a particle M performing a resultant motion. Let the relative displacement of the particle along its path AB in the time interval $\Delta t = t_1 - t$ be specified by the vector $\overline{MM'}$ (Fig. 27 a). In the same time interval, the curve AB moving together with the moving axes $Oxyz$ not shown in Fig. 26 a, occupies a new position A_1B_1. Simultaneously, the point M on curve AB,



a) *Fig. 27* *b)*

with which the particle M is coincident at time t performs a transport displacement. As a result of these displacements particle M will occupy a position M_1 its absolute displacement in the time interval Δt being From the vector triangle $MM''M_1$ we have:

$$\overline{MM_1} = \overline{MM''} + \overline{M''M_1}.$$

Dividing the equation by Δt and passing to the limit, we obtain:

$$\lim_{\Delta t \to 0} \frac{\overline{MM_1}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{MM''}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\overline{M''M_1}}{\Delta t}.$$

By definition

$$\lim_{\Delta t \to 0} \frac{\overline{MM_1}}{\Delta t} = \overline{V}_a \qquad \lim_{\Delta t \to 0} \frac{\overline{MM''}}{\Delta t} = \overline{V}_{tr}.$$

As for the last component, since at $\Delta t \to U$ curve A_1B_1 tends to coincide with curve AB, in the limit we have:

$$\lim_{\Delta t \to 0} \frac{\overline{M''M_1}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{MM'}}{\Delta t} = \overline{V}_{rel}.$$

As a result we obtain:

$$\overline{V}_a = \overline{V}_{rel} + \overline{V}_{tr}. \qquad (34)$$

Vectors $\bar{V}_a$, $\bar{V}_{rel}$ and $\bar{V}_{tr}$ are tangential to the respective paths (Fig. 27, b).

Thus, we have proved the following theorem of the composition of velocities: in resultant motion the absolute velocity of a point is the geometrical sum of the relative velocity and the transport velocity.

The construction in Fig. 27 b is called the parallelogram of velocities.

The absolute velocity in magnitude is equal:

$$V_a = \sqrt{V_{rel}^2 + V_{tr}^2 + 2V_{rel}V_{tr}\cos\alpha}, \tag{35}$$

where α is an angle between the directions of velocities $\bar{V}_{rel}$ and $\bar{V}_{tr}$.

4.3. Velocity and Acceleration of a Point of a Body Having one Fixed Point

Consider some body having one fixed point *0* (Fig. 28). Let us determine the velocity vector and acceleration vector of any point *M* of this body.

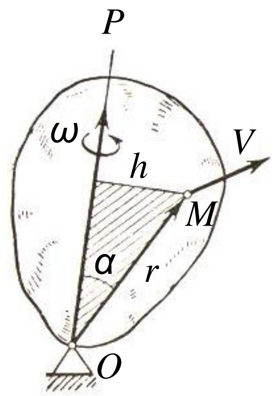


Fig.28

Let the vector of angular velocity of a body at considered instant be $\bar{\omega}$ .Consider the vector product $\bar{\omega}\times\bar{r}$ where $\bar{r}$ is the radius vector from the fixed point *O* to the point *M*. The absolute value of the product is

$$|\bar{\omega}\times\bar{r}| = \omega r\sin\alpha = \omega h.$$

vectors $\bar{\omega}\times\bar{r}$ and $\bar{V}$, it will be readily observed, have the same direction and dimension. Consequently,

$$\bar{V} = \bar{\omega}\times\bar{r}. \tag{36}$$

Hence, the velocity vector for any point *M* of a body is equal to the vector product of the angular velocity of that body and the radius vector of the point.

Now determine the acceleration of point *M* from Eq. (36) differentiating with respect to time we have:

$$\bar{a} = \frac{d\bar{V}}{dt} = \frac{d\bar{\omega}}{dt}\times\bar{r} + \bar{\omega}\times\frac{d\bar{r}}{dt}.$$

But $\frac{d\bar{\omega}}{dt} = \bar{\varepsilon}$ and $\frac{d\bar{r}}{dt} = \bar{V}$, therefore,

$$\bar{a} = \bar{\varepsilon}\times\bar{r} + \bar{\omega}\times\bar{V} \tag{37}$$

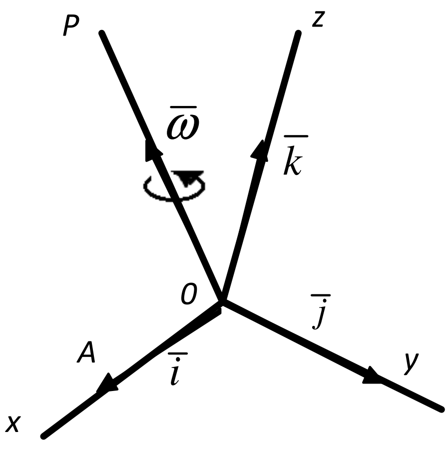


Fig. 29

In some problems of mechanics, particularly in resultant or absolute motion of a particle moving axes *Oxyz* are used. When the axes are in translational motion, the unit vectors $\bar{i}$, $\bar{j}$, $\bar{k}$ remain constant. However, if the trihedron *Oxyz* in Fig. 29 rotates about an axis *OP*, the unit vectors cease to be constants as their directions change with time. In this case, to calculate the derivative of a vector, one must know the derivatives of the unit vectors $\bar{i}$, $\bar{j}$, $\bar{k}$. Unit vector $\bar{i}$ can be treated as the radius vector $\bar{r}_A = \bar{i}$ of a point A on the axis x at unit distance from the origin O. Then,

$$\frac{d\bar{i}}{dt} = \frac{d\bar{r}_A}{dt} = \bar{V}_A.$$

But according to Eq. (36), $\bar{V}_A = \bar{\omega} \times \bar{r}_A = \bar{\omega} \times \bar{j}$ where $\bar{\omega}$ is the angular velocity of the rotation about axis OP. Similar relationships are obtained for the derivatives of $\bar{j}$ and $\bar{k}$, and finally we obtain:

$$\frac{d\,\bar{i}}{dt} = \bar{\omega} \times \bar{i}\,, \quad \frac{d\,\bar{j}}{dt} = \bar{\omega} \times \bar{j}\,, \quad \frac{d\,\bar{k}}{dt} = \bar{\omega} \times \bar{k}\,, \tag{38}$$

4.4. Composition of Accelerations

Let us determine the dependence between the relative and transport accelerations of a particle.

Prom Eq. (34) we obtain:

$$\bar{a}_A = \frac{d\bar{V}_A}{dt} = \frac{d\,\bar{V}_{rel}}{dt} + \frac{d\bar{V}_{tr}}{dt}. \tag{39}$$

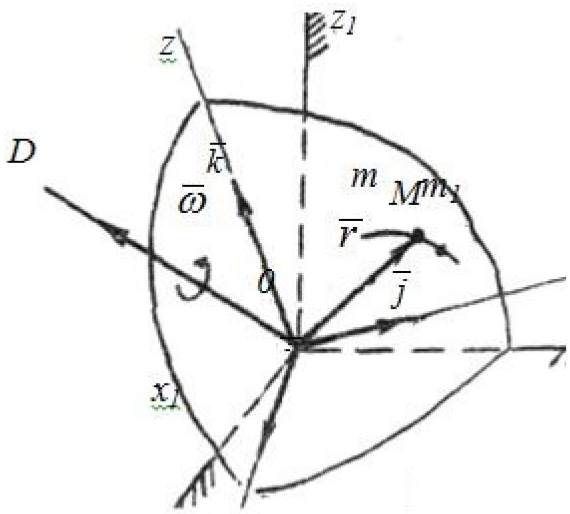


Fig. 30

Let us compute the derivatives in the right-hand side of the equation which, as we shall see, are in general case not equal to $\bar{a}_{rel}$ and $\bar{a}_{tr}$.

Let the position of particle M with respect to the moving axes $Oxyz$ in Fig. 30 be given by its x, y, z coordinates. Since in computing $\bar{V}_{rel}$ and $\bar{V}_{tr}$ the motion of the moving axes is disregarded (they can be assumed fixed), the projections of vectors $\bar{V}_{rel}$ and $\bar{a}_{rel}$ on axes $Oxyz$ in any transport motion are then given by the Eqs. (8) and (10).

Consequently, $\bar{V}_{rel} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k}$, $\quad \bar{a}_{rel} = \ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k}$. (40)

Assume that the angular velocity of the motion of transport is $\bar{\omega}$. In this case axis OD may either be fixed or it may be the instantaneous axis of rotation (when point *0* is fixed; see p.4.3). In both cases the unit vectors are not constant as they change their directions. Therefore we obtain from Eqs.(40):

$$\frac{d\overline{V}_{rel}}{dt}=\left(\ddot{x}\overline{i}+\ddot{y}\overline{j}+\ddot{z}\overline{k}\right)+\left(\dot{x}\frac{d\overline{i}}{dt}+\dot{y}\frac{d\overline{j}}{dt}+\dot{z}\frac{d\overline{k}}{dt}\right)=\overline{a}_{rel}+\overline{a}_{1},$$

where $\overline{a}_1$ denotes the second bracket in the right-hand side of the equation. Calculating $\overline{a}_1$ with the help of Eqs.(38) we obtain:

$$\overline{a}_1=\dot{x}\left(\overline{\omega}\times\overline{i}\right)+\dot{y}\left(\overline{\omega}\times\overline{j}\right)+\dot{z}\left(\overline{\omega}\times\overline{k}\right)=\overline{\omega}_x\left(\dot{x}\overline{i}+\dot{y}\overline{j}+\dot{z}\overline{k}\right)=\overline{\omega}\times\overline{V}_{rel},$$

and finally,

$$\frac{d\overline{V}_{rel}}{dt}=\overline{a}_{rel}+\overline{a}_1,\quad \text{where}\quad \overline{a}_1=\overline{\omega}\times\overline{V}_{rel}. \tag{41}$$

In this equation the quantity $\overline{a}_{rel}$ takes into account the change in vector $\overline{V}_{rel}$ only in the relative motion, and the new member $\overline{a}_1$ takes into account the change of vector $\overline{V}_{rel}$ in its rotation together with trihedron *Oxyz* around the axis *OP*, i.e., in the motion of transport.

Furthermore, in rotational motion the velocity and acceleration of any point fixed with respect to the axes *Oxyz* are determined by Eq. (36) and (37), the same as for the points of a rigid body. But $\overline{V}_{tr}=\overline{V}_m$ and $\overline{a}_{tr}=\overline{a}_m$, hence Eqs. (36) and (37) yield

$$\overline{V}_{tr}=\overline{\omega}\times\overline{r},\qquad \overline{a}_{tr}=\left(\overline{\varepsilon}\times\overline{r}\right)+\left(\overline{\omega}\times\overline{V}_{tr}\right), \tag{42}$$

where $\overline{r}$ is the radius vector of point *m* coincident at the given instant with the radius vector of the moving particle *M*. Hence,

$$\frac{d\overline{V}_{tr}}{dt}=\left(\frac{d\overline{\omega}}{dt}\times\overline{r}\right)+\left(\overline{\omega}\times\frac{d\overline{r}}{dt}\right),$$

here $\frac{d\overline{r}}{dt}=\overline{V}_a=\overline{V}_{rel}+\overline{V}_{tr}$, and besides $\frac{d\overline{\omega}}{dt}=\overline{\varepsilon}$. Hence,

$$\frac{d\overline{V}_{tr}}{dt}=\left(\overline{\varepsilon}\times\overline{r}\right)+\left(\overline{\omega}\times\overline{V}_{tr}\right)+\left(\overline{\omega}\times\overline{V}_{rel}\right).$$

From this equation, taking into account the second of Eqs.(42), we obtain:

$$\frac{d\overline{V}_{tr}}{dt}=\overline{a}_{tr}+\overline{a}_2,\quad \text{where}\quad \overline{a}_2=\overline{\omega}\times\overline{V}_{rel} \tag{43}$$

The quantity $\overline{a}_{tr}$ takes into account the change in vector $\overline{V}_{tr}$ only in the motion of transport, because it is computed as the acceleration of point *m*, fixed in the reference frame *Oxyz*. But the new member $\overline{a}_2$ takes into account the change in vector $\overline{V}_{tr}$ that occurs in the relative motion of particle *M*, since as a result of that motion *M* moves from position *m* to a new position m_1 where the value of $\overline{V}_{tr}$ is different.

Now substituting the quantities (41) and (43) into Eq. (39), we have:

$$\overline{a}_a = \overline{a}_{rel} + \overline{a}_{tr} + \overline{a}_1 + \overline{a}_2. \qquad (44)$$

Let us introduce the notation:

$$\overline{a}_{cor} = \overline{a}_1 + \overline{a}_2 = 2\left(\overline{\omega} \times \overline{V}_{rel}\right). \qquad (45)$$

The quantity $\overline{a}_{cor}$ which characterizes the rate of change of the vector of relative velocity in the motion of transport and the rate of change of the vector of the transport velocity in the relative motion is called the supplementary, or Coriolis, acceleration of the particle. Then, from Eqs. (44) and (45) we obtain:

$$\overline{a}_a = \overline{a}_{rel} + \overline{a}_{tr} + \overline{a}_{cor}. \qquad (46)$$

Eq. (46) expresses the Coriolis theorem:

the absolute acceleration of a particle is equal to the geometrical sum of three accelerations: the relative acceleration, which characterizes the time rate of change of the relative velocity in the relative motion, the transport acceleration, which characterizes the time rate of change of the transport velocity in the transport motion, and the Coriolis acceleration, which characterizes the time rate of change of the relative velocity in the transport motion and of the transport velocity in the relative motion.

4.5. Calculation of relative, Transport, and Coriolis Accelerations

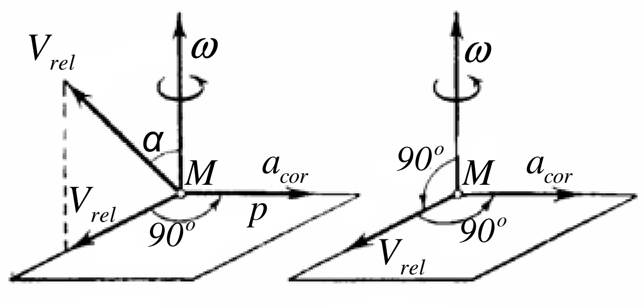
We examined the question of computing the relative and transport accelerations of a particle in proving the theorem. These quantities are determined, according to the known equations of Kinematics. Relative acceleration is computed by the conventional methods of particle Kinematics. In calculating $\overline{a}_{tr}$ the relative motion of the particle can be disregarded, consequently $\overline{a}_{tr}$ is computed as the acceleration of a point belonging to a certain rigid body fixed relative to the reference frame *Oxyz* and moving together with it, i.e. by the methods of rigid-body Kinematics.

The Coriolis acceleration is calculated from Eq. (45):

$$\overline{a}_{cor} = 2\overline{\omega} \times \overline{V}_{rel}, \qquad (47)$$

where $\overline{\omega}$ is the angular velocity of the motion of transport.

Thus, the Coriolis acceleration of a particle is equal to the double vector product of the angular velocity of the motion of transport and the relative velocity of the particle.



a) Fig. 31 b)

If the angle between the vectors $\overline{V}_{rel}$ and $\overline{\omega}$ is α then in magnitude $a_{cor} = 2\omega V_{rel} \sin\alpha.$ (48)

The vector $\overline{a}_{cor}$ is of the same sense as the vector $\overline{\omega} \times \overline{V}_{rel}$, i.e., normal to the plane through vectors $\overline{\omega}$ and $\overline{V}_{rel}$ in the direction from which a counterclockwise rotation would be seen to carry vector

$\bar{\omega}$ into vector $\overline{V}_{rel}$ through the smaller angle (Fig. 31, a). It can also be seen that the direction of vector $\bar{a}_{cor}$ can be obtained by projecting vector $\overline{V}_{rel}$ on plane *P*, which is normal to $\bar{\omega}$, and turning that projection through 90^0 in the direction of the rotation of transport.

If the relative path is a plane curve, moving in its plane, then angle α = 90^0 (Fig.31, b) and in magnitude $a_{cor} = |2\omega \cdot V_{rel}|$. It can be seen from Fig. 31, b that in this case the direction of $\bar{a}_{cor}$ can be obtained by turning the vector of the relative velocity $\overline{V}_{rel}$ through 90^0 in the direction of the rotation of transport.

From Eq. (48) we see that the Coriolis acceleration is zero when:

1) $\omega = 0$, if the motion of transport is translational or if the angular velocity of the rotation of transport becomes zero at a given instant;
2) $V_{rel} = 0$, if there is no relative motion or if the relative velocity becomes zero at a given instant;
3) angle $\alpha = 0^0$, $\alpha = 180^0$, i.e., if the relative motion is parallel to the axis of the rotation of transport or if vector $\overline{V}_{rel}$ parallel to that axis at a given instant.